GROUP THEORY 2024 - 25, SOLUTION SHEET 3

Exercise 1. *Normal subgroups and Group Quotients*
See algebraic structures lecture notes.

Exercise 2. *First isomorphism theorem*
See algebraic structures lecture notes.

Exercise 3. *Correspondence Theorem and the Third Isomorphism Theorem*
See algebraic structures lecture notes.

Exercise 4. *Second isomorphism theorem*
See algebraic structures lecture notes.

Exercise 5. *Equivalence of definitions of group actions: Very important! To remember and use in practice!*
We construct a map

$$f : \{\Phi : G \to \mathrm{Bij}(X) \mid \Phi \text{ is an action}\} \cong \{\cdot : G \times X \to X | \text{ (1) \& (2) hold}\}$$

given by

$$f : \Phi \mapsto \cdot_\Phi$$

where $\cdot_\Phi : G \times X \to X$ is given by $g \cdot_\Phi x = \Phi(g)(x)$. Let us check that this set map is well defined, i.e. that $\cdot_\Phi$ satisfies (1) and (2):

(1) $e_G \cdot_\Phi x = \Phi(e_G)(x) = x$, for all $x \in X$, by the fact that Φ is an action.
(2) $g \cdot_\Phi (h \cdot_\Phi x) = g \cdot_\Phi \Phi(h)(x) = \Phi(g)(\Phi(h)(x)) = \Phi(gh)(x) = (gh) \cdot_\Phi x$, for all $g, h \in G$ and $x \in X$, again by the fact that Φ is an action.

Similarly, we now construct a map in the other direction

$$g : \{\cdot : G \times X \to X | \text{ (1) \& (2) hold}\} \cong \{\Phi : G \to \mathrm{Bij}(X) \mid \Phi \text{ is an action}\}$$

given by

$$\cdot \mapsto \Phi.$$

where $\Phi. : G \to Bij(X)$ is given by $\Phi.(g)(x) = g \cdot x$. Let us check that this set map is well defined, i.e. that $\Phi.$ is an action. According to the lecture notes, it suffices to prove multiplicativity and that each $\Phi.(g) : X \to X$ is a bijection:

(1) Multiplicativity follows from that of $\cdot$: $\Phi.(gh)(x) = (gh) \cdot x = g \cdot (h \cdot x) = \Phi.(g)(\Phi.(h)(x))$ for all $g, h \in G$ and $x \in X$, so $\Phi.(gh) = \Phi.(g) \circ \Phi.(h)$.

(2) Let us show that for all $g \in G$, $\Phi.(g)$ is a bijection. For the surjectivity, take $x \in X$ arbitrary. Then $\Phi.(g)(g^{-1} \cdot x) = g \cdot (g^{-1} \cdot x) = (g \cdot g^{-1}) \cdot x = e_G \cdot x = x$. For injectivity, take $x, y \in X$ with $g \cdot x = \Phi.(g)(x) = \Phi.(g)(y) = g \cdot y$. By taking $g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (g \cdot y)$ we obtain $x = y$.

It is straightforward to check that $f \circ g$ and $g \circ f$ are the identities on the respective sets, so we have our desired bijection.

Exercise 6. To prove that the action $\Phi : G \to Bij(G/H)$ given by $\Phi_g(aH) = gaH$ is not faithful, we need to find $g \neq g' \in G$ such that $\Phi_g = \Phi_{g'}$. As H has at least two elements, take g and g' to be any such different elements of H. Let us show that $\Phi_g(aH) = \Phi_{g'}(aH)$, for all $a \in G$. Observe that $\Phi_g(aH) = \Phi_{g'}(aH) \iff gaH = g'aH \iff a^{-1}g'^{-1}ga \in H$, but this is true for all $a \in G$, because $g'^{-1}g \in H$ by construction and H is normal.

Exercise 7. Let $g \in G$, clearly if $g \in H$ then $gHg^{-1} = H$. If $g \notin H$, then since the index of H in G is two, we have that

$$G/H = \{H, gH\} \text{ and } H\backslash G = \{H, Hg\}.$$

This implies that $gH = Hg$ as sets. It follows that $gHg^{-1} = H$. □

Exercise 8. *Some properties of cosets useful in practice*

(1) We have the following equivalences

$$\begin{aligned} gH = g'H &\iff \exists h \in H \text{ such that } g' = gh \\ &\iff \exists h \in H \text{ such that } g^{-1}g' = h \\ &\iff g^{-1}g' \in H. \end{aligned}$$

(2) You showed in the lectures that the cosets form a partion of G, hence they must coincide or be disjoint.

(3) Suppose $gH \cap g'K \neq \emptyset$. This means there exists an element $x \in G$ such that $x \in gH$ and $x \in g'K$. Thus, we have:

$$x = gh_1 = g'k_1$$

for some $h_1 \in H$ and $k_1 \in K$. Rearranging this equation gives:

$$g^{-1}g' = h_1k_1^{-1} \tag{1}$$

We are going to show by double inclusion that $gH \cap g'K = gh_1(H \cap K)$.
Suppose first that $y \in gH \cap g'K$. We can write:

$$y = gh_2 = g'k_2$$

for some $h_2 \in H$ and $k_2 \in K$. Rearranging the equation and using (1) we obtain:

$$h_2 = g^{-1}g'k_2 = (h_1k_1^{-1})k_2,$$

From this we can deduce that $h_1^{-1}h_2 = k_1^{-1}k_2 \in K \cap H$. Hence, we can write

$$y = gh_2 = gh_1(k_1^{-1}k_2) \in gh_1(K \cap H).$$

This prove the first inclusion. The second inclusion is direct.

Exercise 9. (1) Let $\varphi : X \to Y$ be a G-set isomorphism. Assume X is transitive. Let $y_1, y_2 \in Y$. Since φ is a bijection, there exist $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Since X is transitive, there exists $g \in G$ such that $g \cdot x_1 = x_2$. Applying the isomorphism φ to both sides, we get:

$$g \cdot y_1 = g \cdot \varphi(x_1) = \varphi(g \cdot x_1) = \varphi(x_2) = y_2.$$

Hence, Y is transitive. If Y is transitive, show that $\varphi^{-1} : Y \to X$ is a G-set homomorphism and apply the same reasoning to φ^{-1} to show that X is transitive.

(2) Immediate.

(3) We construct a function

$$\begin{aligned}\{\text{Conjugacy classes of subgroups } H \leq G\} &\to \mathcal{X}/\sim \\ [H] &\mapsto [G/H]\end{aligned}$$

where G/H is endowed with the usual G-action. This action is clearly transitive. Moreover this map is well defined because if $[H] = [H']$, meaning that H and H' are conjugate, then the two G-sets G/H and G/H' are isomorphic by exercise 3 of week 2. Hence they are in the same isomorphism class and so they define the same element $[G/H] = [G/H']$ of $\mathcal{X}/\sim$.

We show that this function is bijective. It is injective because if H,H' are subgroups such that G/H and G/H' are isomorphic as G-sets, then H and H' are conjugate by exercise 3 of week 2.

To show that it is surjective, let $G \curvearrowright X$ be a transitive G-set. Choose a point $x \in X$ and consider its stabilizer $H = \mathrm{Stab}_G(x)$. By the orbit-stabilizer theorem there is a bijection between X and G/H given by:

$$\begin{aligned}f : G/H &\to X \\ gH &\mapsto g \cdot x.\end{aligned}$$

This map is G-equivariant since

$$f(a \cdot gH) = f(agH) = ag \cdot x = a \cdot (g \cdot x) = a \cdot f(gH)$$

and thus $G/H \cong X$ as G-sets. Therefore, any transitive G-set is isomorphic to G/H for some subgroup H of G. This shows surjectivity.

(4) (a) $G = \mathbb{Z}/4\mathbb{Z}$. Its subgroups are:
- $\langle 0 \rangle$ (trivial subgroup)
- $\langle 2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\mathbb{Z}/4\mathbb{Z}$

None of these subgroups are conjugate (since $\mathbb{Z}/4\mathbb{Z}$ is abelian), so we have three distinct conjugacy classes. Thus, there are three isomorphism classes of transitive $\mathbb{Z}/4\mathbb{Z}$-actions corresponding to these subgroups.

(b) $G = \mathbb{Z}/8\mathbb{Z}$. Its subgroups are:
- $\langle 0 \rangle$
- $\langle 4 \rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\langle 2 \rangle \cong \mathbb{Z}/4\mathbb{Z}$
- $\mathbb{Z}/8\mathbb{Z}$

Again, none of these subgroups are conjugate, so there are four distinct isomorphism classes of transitive $\mathbb{Z}/8\mathbb{Z}$-actions.

(c) $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Its subgroups are:
- $\langle (0,0) \rangle$ (trivial subgroup)
- $\langle (1,0) \rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\langle (0,1) \rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\langle (1,1) \rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

All these subgroups are distinct and not conjugate (as $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is abelian), so there are five distinct isomorphism classes of transitive actions.

(d) $G = S_3$. Its subgroups are:
- $\langle e \rangle$ (trivial subgroup)
- Three subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (generated by transpositions)
- $\langle (123) \rangle \cong \mathbb{Z}/3\mathbb{Z}$
- S_3

The three subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}$ are conjugate to each other, and the rest are not conjugate. Hence, we have four distinct isomorphism classes of transitive S_3-actions.